Avoiding saddle points in nonsmooth optimization

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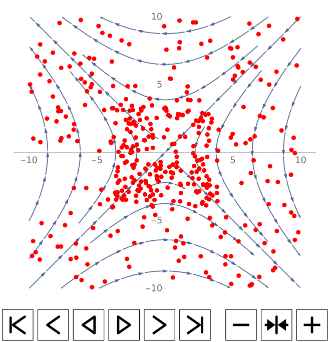
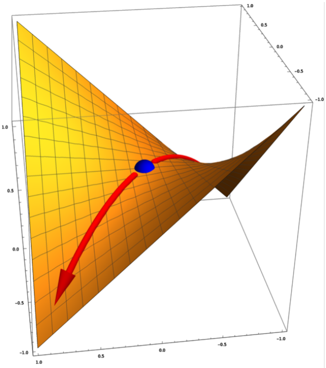
Saddle point avoidance

Recent Realization:

Simple algorithms for minimizing C^2 functions avoid all strict saddle points, when randomly initialized.[1]

- Simple algorithms: Gradient descent (GD), coordinate descent....
- *Strict* saddle points: Critical points that have negative curvature.





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For a wealth of estimation and learning problems, all spurious critical points are strict saddles and therefore avoidable!

(Sun-Qu-Wright '15-'18, Ge-Lee-Ma '16, Bhojanapalli-Neyshabur-Srebro '16, Ge-Jin-Zheng '17...)

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This talk:

Do first-order methods avoid "strict saddles" of nonsmooth functions?

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Weak convexity: an amenable problem class

$$\underset{x\in\mathbb{R}^d}{\text{minimize}}\ F(x)$$

Running assumption: weak convexity

$$F(\cdot)+\frac{\rho}{2}\|\cdot\|^2 \qquad \text{is convex.}$$

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Main example: $\boxed{(\text{convex}) \circ (\text{smooth})}$

$$h(c(x))$$

h is convex and L-Lipschitz; c is smooth with ℓ-Lipschitz Jacobian ($\rho = L\ell$)

(Fletcher '80, Powell '83, Burke '85, Wright '90, Lewis-Wright '08, Cartis-Gould-Toint '11,...)

Example: Low-rank Matrix Estimation

Set-up: Fix rank r matrix $M_\sharp \succeq 0$ and observe measurements

$$\langle A_i, M_\sharp \rangle \approx b_i \qquad \forall i = 1, \ldots, m.$$

Goal: Recover $M_\sharp$ from b_i

[3] Burer-Monteiro '01

[4] Candes-Tao '05, Chen-Chi-Goldsmith '13

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Natural Nonconvex Penalty Formulation:[3]

$$\min_{M \in \mathbb{R}^{d \times d}} \; |||\mathcal{A}(M) - b||| \qquad \text{subject to: } M \text{ is rank} \leq r$$

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$$M = XX^T \qquad X \in \mathbb{R}^{d \times r}$$

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- ℓ_2: Gaussian A_i/Gaussian noise, leads to smooth problems.
- ℓ_1: structured A_i/sparse corruption, leads to nonsmooth problems.

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First-order methods for nonsmooth problems

Common iterative methods take form

$$x_{t+1} = \arg\min_{y} F_{x_t}(y)$$

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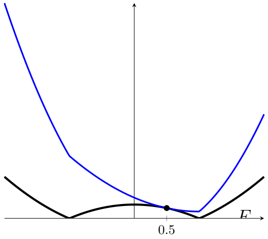
Example: Proximal point



$$F_{x_t}(y) = F(y) + \frac{1}{2\eta}\|y - x_t\|^2$$

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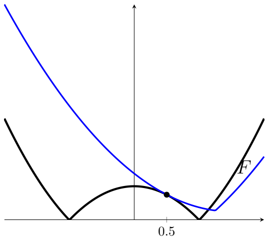
Example: Proximal linear (for $F = h \circ c$)



$$F_{x_t}(y) = h(c(x_t) + \nabla c(x_t)(y - x_t)) + \frac{1}{2\eta}\|y - x_t\|^2$$

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Example:

Algorithm	Objective F	Update function $F_x(y)$
Prox-point	$F(x)$	$F(y) + \frac{1}{2\eta}\|y - x\|^2$
Prox-linear	$h(c(x)) + r(x)$	$h(c(x) + \nabla c(x)(y - x)) + r(y) + \frac{1}{2\eta}\|y - x\|^2$
Prox-gradient	$f(x) + r(x)$	$f(x) + \langle \nabla f(x), y - x\rangle + r(y) + \frac{1}{2\eta}\|y - x\|^2$

Table: h is convex and Lipschitz, r is weakly convex, and f and c are C^2-smooth.

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Equivalent when F is C^2.

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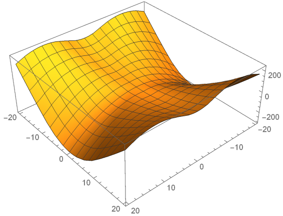
Negative curvature is not enough even for C^1 functions



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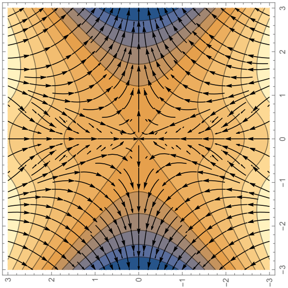

(b) Flow $\dot{\gamma} = -\nabla F(\gamma)$

$$F(x, y) = \textbf{Moreau}\{(|x| + |y|)^2 - 2y^2\}$$

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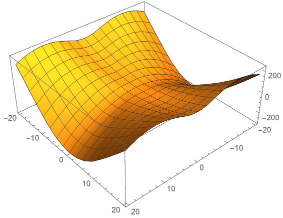

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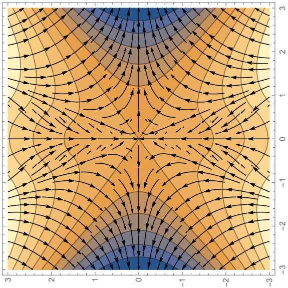

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Problem: do not reach y axis fast enough to benefit from curvature!

An extra ingredient: sharpness

Idea: Require F to grow **sharply** away from axis:

$$\inf\{\|\nabla F(x,y)\|: \text{ for } (x,y) \text{ off of } y \text{ axis}\} > 0$$

Benefit: Ensures grad. flow aims towards axis with (at least) constant speed.

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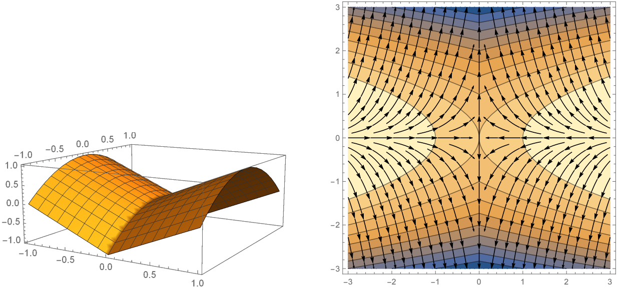

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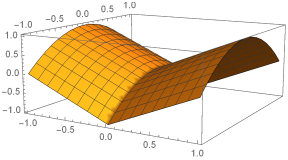

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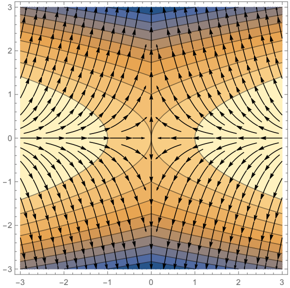

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Question: How to generalize?

The active manifold

Idea: Replace axis with "active manifold" of smoothness.

(Wright '93, Lemaréchal-Oustry-Sagastizábal '96, Bonnans-Shapiro '00, Lewis '03, Drusvyatskiy-Lewis '14...)

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Defn: Critical point lies on C^2-smooth "active manifold $\mathcal{M}$":

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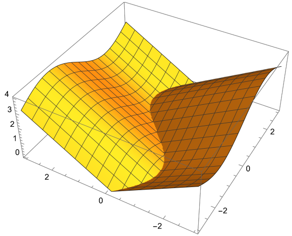

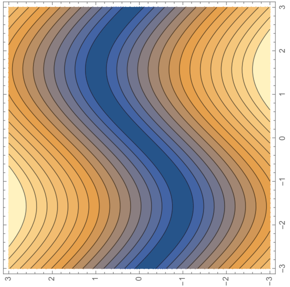

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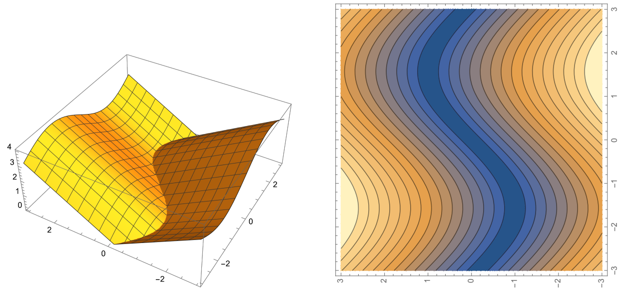

Question: What about curvature?

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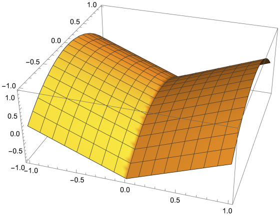
Putting it all together: the active strict saddle property



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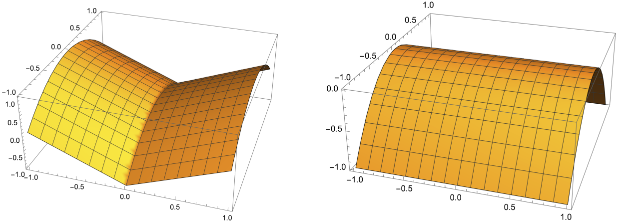

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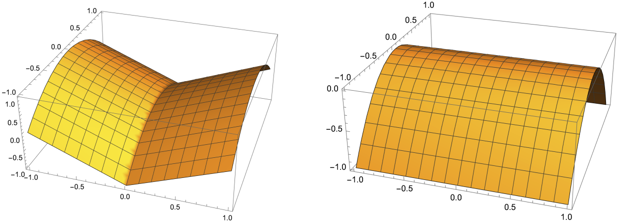

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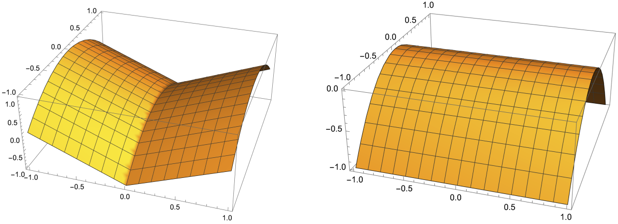

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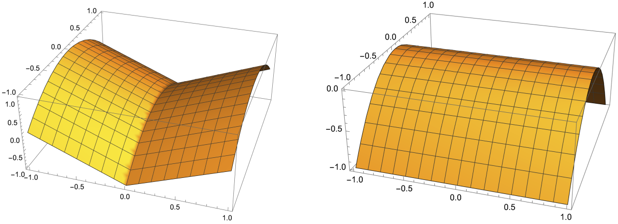
1. F admits active manifold $\mathcal{M}$ containing $\bar{x}$.
2. The smooth extension $F \circ P_{\mathcal{M}}$ has a strict saddle point at $\bar{x}$:

$$\lambda_{\min}(\nabla^2(F \circ P_{\mathcal{M}})(\bar{x})) < 0.$$



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Although it may seem stringent, this property is **generic**:

Theorem (Drusvyatskiy-Ioffe-Lewis '16, D-Drusvyatskiy '19)

If F is **semi-algebraic** *and weakly convex, then for full Lebesgue measure set of perturbations $v \in \mathbb{R}^d$ every critical point of*

$$F_v(x) = F(x) - \langle v, x \rangle$$

is either an active strict saddle or a **local minimizer**.

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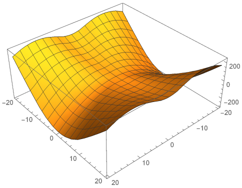

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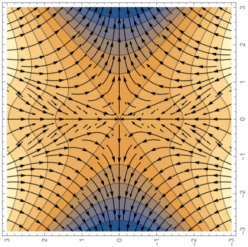

(b) Flow $\dot{\gamma} = -\nabla F(\gamma)$

Example is Highly Unstable: small linear tilts do not exhibit this behavior!

Question: Do the three proximal methods avoid active strict saddles?

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Key: view algorithms

$$x_{t+1} = \arg\min_y F_{x_t}(y),$$

as fixed-point iteration of well-behaved operator T.[6]

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Recipe for smooth functions

Fixed point iteration

$$x_{t+1} = T(x_t)$$

Recipe for smooth functions

Fixed point iteration

$$x_{t+1} = T(x_t) \qquad [\text{Grad descent is } T = I - \eta \nabla F]$$

Recipe for smooth functions

Fixed point iteration

$$x_{t+1} = T(x_t) \qquad\qquad [\text{Grad descent is } T = I - \eta\nabla F]$$

Recipe:

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$$\nabla F(\bar{x}) = 0 \qquad \text{and} \qquad \lambda_{\min}(\nabla^2 F(\bar{x})) < 0$$

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Important: Argument requires that T is local diffeomorphism.

Beyond gradient descent

To apply argument, need

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Focus on Local Smoothness, since other calculation complex.

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Surprising: Function F is nonsmooth, yet S is C^1 around strict saddles. Why?

[7] Lemaréchal-Sagastizábal '97

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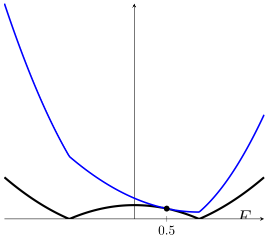
Example: Prox-point



$$F_{x_t}(y) = F(y) + \frac{1}{2\eta}\|y - x_t\|^2$$

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Then Weak convexity + classical perturbation theory $\implies$ S is C^1 near $\bar{x}$.[7]

[7] Lemaréchal-Sagastizábal '97

Avoiding active strict saddles

Proof extends to the three methods:

Algorithm	Objective F	Update function $F_x(y)$
Prox-point	$F(x)$	$F(y) + \frac{1}{2\eta}\|y-x\|^2$
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Table: h is convex and Lipschitz, r is weakly convex, and f and c are C^2-smooth.

Avoiding active strict saddles

Theorem: (Local smoothness, D-Drusvyatskiy '19)

Around each active strict saddle $\bar{x}$ of F, the iteration mapping

$$S(x) = \arg\min_y F_x(y),$$

is C^1 and the Jacobian $\nabla S(\bar{x})$ has a real EigVal strictly greater than 1

Proof more interesting/surprising for prox-gradient and prox-linear.

Avoiding active strict saddles

Problem: S may not be Local diffeomorphism

Avoiding active strict saddles

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Easy solution: Add damping

$$T = (1 - \lambda)I + \lambda S.$$

Avoiding active strict saddles

Corollary: (Random initialization, D-Drusvyatskiy '19)

Randomly initialized three methods with small damping

$$x_{t+1} = (1-\lambda)x_t + \lambda S(x_t),$$

locally escape active strict saddles.

Globalization:

- Results hold globally when S is Lipschitz (prox-point, prox-gradient)
- **Open Problem:** Is prox-linear update globally Lipschitz?

Beyond proximal methods

Limitation of result: Only applies to three "proximal methods."

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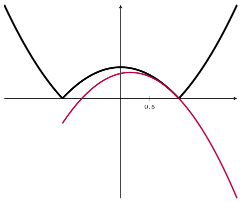
Alternative: subgradient method

The subdifferential of a weakly convex function

Fact: For any $F\colon \mathbb{R}^d \to \mathbb{R}$, have equivalence:

- F is ρ-weakly convex
- Subgradient inequality: $\forall x \exists v_x$ satisfying

$$F(y) \geq F(x) + \langle v_x, y - x\rangle - \frac{\rho}{2}\|y - x\|^2$$



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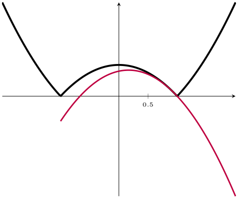

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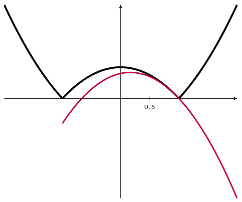

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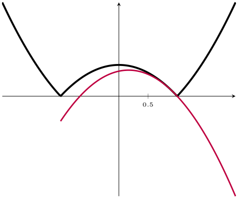

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Fermat's rule: If $\bar{x}$ is a local minimizer of F then

$$0 \in \partial F(\bar{x}).$$

Alternative: the subgradient method

Idea: At time t

[8] Bolte-Pauwels '19-'20

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Idea: At time t

1. **"Linearize F:"** choose $v_t \in \partial F(x_t)$ and form

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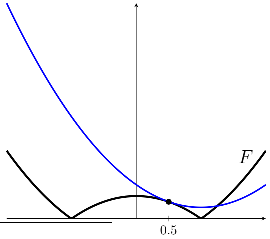

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2. Can often replace v_t with result of auto-differentiation procedure.[8]

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Extension: Subgradient method

Question: Does subgradient method avoid active strict saddle points?

$$x_{t+1} \in x_t - \alpha_t \partial F(x_t)$$

[9] D-Drusvyatskiy-Jiang '21

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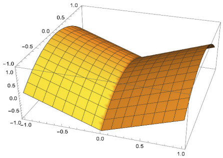
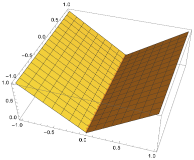
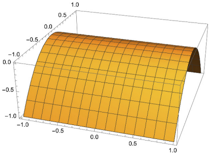
Key: "orthogonal decomposition" of trajectory.

[9] D-Drusvyatskiy-Jiang '21

$\mathcal{VU}$ decomposition[10]

 $=$  $+$ 

$F \qquad\qquad F_{\mathcal{V}} \qquad\qquad F_{\mathcal{U}}$

[10] Mifflin-Sagastizábal '05

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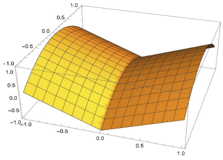

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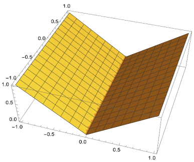

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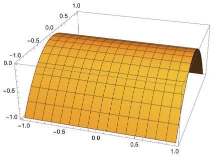

$F \qquad F_{\mathcal{V}} \qquad F_{\mathcal{U}}$

Decompose trajectory:

[10] Mifflin-Sagastizábal '05

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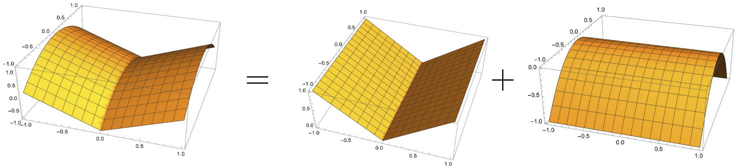

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Decompose trajectory:

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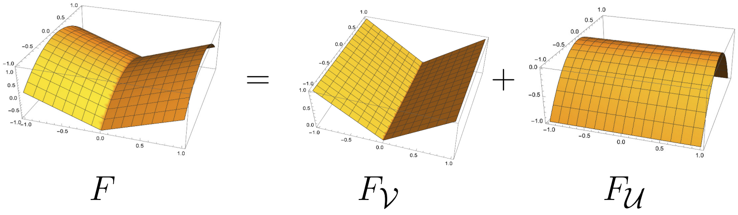

Decompose trajectory:

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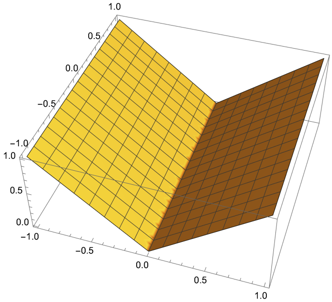
[10] Mifflin-Sagastizábal '05

The two regularity assumptions

1. **Aiming:** Negative subgradients aim towards manifold:

$$\text{Sharpness} \implies \langle \widetilde{\nabla} F_{\mathcal{V}}(x_t), x_t - P_{\mathcal{M}}(x_t) \rangle \geq \mu \, \mathrm{dist}(x_t, \mathcal{M})$$



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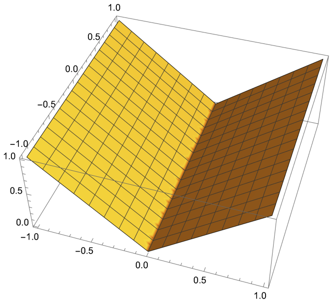

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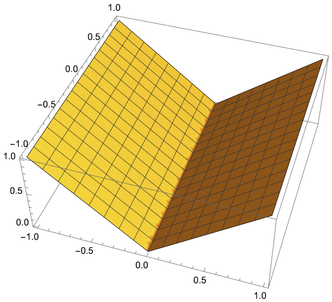


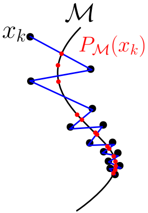
Prevalent: true generically for weakly convex semialgebraic problems.

The two pillars

The two pillars: For a wide class of problems

- Subgradient method quickly approaches the active manifold:

$$\mathrm{dist}(x_t, \mathcal{M}) = O(\alpha_t).$$



(a) Quickly approach manifold

The two pillars

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- Subgradient method quickly approaches the active manifold:

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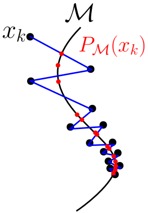
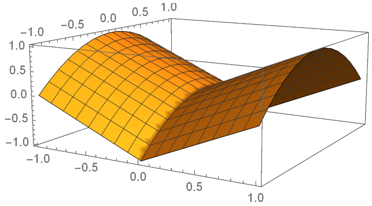
- The shadow $y_t = P_{\mathcal{M}}(x_t)$ forms inexact Riemannian gradient sequence:

$$y_{t+1} = y_t - \alpha_t \nabla_{\mathcal{M}} F(y_t) + O(\alpha_t \operatorname{dist}(x_t, \mathcal{M}) + \alpha_t^2).$$



(a) Quickly approach manifold



(b) “Smooth in tangent directions”

The two pillars

The two pillars: For a wide class of problems

- Subgradient method quickly approaches the active manifold:

$$\mathrm{dist}(x_t, \mathcal{M}) = O(\alpha_t).$$

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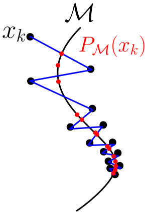

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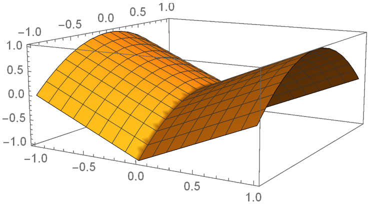

(b) "Smooth in tangent directions"

Conclusion: Get to the manifold quick enough to leverage smoothness of F!

Main result

Due to inexactness, must analyze "perturbed" subgradient method[11]:

$$x_{t+1} \in x_t - \alpha_t(\partial F(x_t) + \nu_t) \qquad \text{where } \nu_t \sim \mathsf{Unif}(B).$$

[11] D-Drusvyatskiy-Jiang '21

[12] Concurrent work: Bianchi-Hachem-Schechtman'21.

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Extensions.

1. **Algorithms:** Proximal/projected subgradient methods.

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Extensions.

1. **Algorithms:** Proximal/projected subgradient methods.
2. **Beyond weak convexity:** Clarke regularity.

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[11] D-Drusvyatskiy-Jiang '21

[12] Concurrent work: Bianchi-Hachem-Schechtman'21.

Thank you!

References

- Proximal methods avoid active strict saddles of weakly convex functions
  D, Drusvyatskiy. *Found. Comput. Math.* `arxiv.org/abs/1912.07146`.
- Subgradient methods near active manifolds: saddle point avoidance, local convergence, and asymptotic normality
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