ORIE 6300 Mathematical Programming I	September 28, 2016

Recitation 6

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Uncapacitated Network Flow Problem [1]

Suppose we have a directed graph $G = (V, E)$ with n nodes and m arcs. Additionally, there is a vector b of supplies available at each node (where $b_i < 0$ corresponds to a demand). Finally, there is a vector of costs c such that for each edge $e \in E$, c_e is the cost of sending one unit of flow along edge e. Assume the graph is connected and $\sum_i b_i = 0$, i.e., the total supply equals the total demand.

Note that the transportation problem discussed last week can be considered a special case of problems under this framework. (Set of supply nodes, $S = \{i \in V : b_i > 0\}$, demand nodes $D = V \setminus S$, $E = \{(i,j) : i \in S, j \in D\}$.)

A vector of flows, $f \in \mathbf{R}^m$ is feasible for this problem if $f \geq 0$ and the flow constraints for each vertex i are satisfied, i.e.:

$$\sum_{(i,j)\in E} f_{(i,j)} - \sum_{(j,i)\in E} f_{(j,i)} = b_i$$

If we define our matrix A by:

$$a_{ik} = \begin{cases} 1 & \text{if } i \text{ is the start node of arc } k \\ -1 & \text{if } i \text{ is the end node of arc } k \\ 0 & \text{otherwise} \end{cases}$$

then our problem fits into a linear programming framework as:

$$\begin{array}{lll} \min & cf & \\ \text{s.t.:} & Af & = b \\ & f & \geq 0. \end{array}$$

Just like with the transportation problem, it is easy to verify that this matrix is not full rank, but by deleting one constraint row arbitrarily, it will be full rank assuming the graph is connected.

Circulations and Cycles

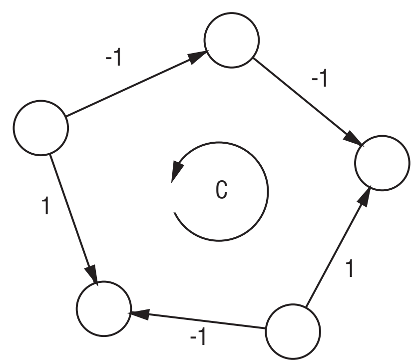
Given a feasible solution f, clearly any direction h that maintains feasibility must satisfy $Ah = 0$. Directions that satisfy this property are called **circulations.** Recall that during the transportation problem, each cycle induced a circulation by choosing each cycle edge to be ± 1 appropriately. This holds for network flow problems as well.

[1] Based on previous notes of Chaoxu Tong

Let C be an **undirected** cycle, in the graph. Let F be the set of forward edges in C and B the set of backwards edges. Then we can define a circulation h^C by:

$$h_e^C = \begin{cases} 1 & e \in F \\ -1 & e \in B \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, this satisfies $Ah = 0$. Finally, the theorem about basic solution from last week holds in this setting.



Claim 1 *If f is a basic solution corresponding to basis B, then B has no cycles.*

Reduced Costs and Network Simplex

Recall the Simplex method maintains a (usually-infeasible) dual solution to compute the reduced costs of the primal variables. Consider the dual of the above network problem:

Primal LP:

min	cf	
s.t.:	$\sum_{(i,j)\in E} f_{(i,j)} - \sum_{(j,i)\in E} f_{(j,i)} = b_i$	$\forall i \in V$
	$f \geq 0$	

Dual LP:

max	wb	
s.t.:	$w_i - w_j \leq c_{(i,j)}$	$\forall (i,j) \in E$

So the dual vector is a set of node variables with the property that $w_i - w_j \leq c_{(i,j)}$ for each edge (i,j). Thus, the reduced costs that we consider are

$$\bar{c}_{(i,j)} = c_{(i,j)} + w_j - w_i.$$

Note that we removed the constraint corresponding to node n in the primal to ensure that A would have full rank. The result is that complementary dual solutions always have w_n fixed arbitrarily. Then, we use the fact that $c_{(i,j)} = w_j - w_i$ for any edge (i,j) in our basis (complementary slackness condition) to compute the rest of the w vector.

Recall the general framework of the simplex method:

1. Start with some basic feasible solution.

2. Compute reduced costs using the complementary dual solution.

3. If the dual is feasible, done. Otherwise, use some negative reduced cost to give a direction of improvement.

4. Use that direction to move to a new basic feasible solution, or to prove that the objective function value is unbounded.

In this model, since calculating the dual solution can be done directly without matrix inversion, we get an enhanced version of the Simplex method by applying the general Simplex principles as follows:

1. Begin with basic feasible solution f corresponding to tree T.

2. Compute the complementary dual vector w by assigning $w_n = 0$, and using $c_{(i,j)} - w_i + w_j = 0$ for edges $(i, j) \in T$. The fact that w_n can be set fixed arbitrarily is proven below.

3. Use the dual vector w to calculate the reduced costs using $\bar{c}_{(i,j)} = c_{(i,j)} - w_i + w_j$. If $\bar{c} \geq 0$, the current solution is optimal. Otherwise, choose some edge e^* with $\bar{c}_{e^*} < 0$ to enter the basis.

4. Find the unique cycle C that contains e^* and edges of T. Push flow around C until some edge $\hat{e}$ has value $f_{\hat{e}} = 0$. If this never happens, h^C is a direction of unbounded improvement.

5. Repeat with the new basis $T \cup \{e^*\} \setminus \hat{e}$.

Proof that complementary dual solutions have w_n can be fixed arbitrarily

Consider the following LP with a redundant constraint and its dual

$$
\begin{array}{lll}
\min & c^x & \\
\text{s.t.:} & Ax & = b \\
 & a^T x & = d \text{ (redundant through linear dependence with Ax = b)} \\
 & x & \geq 0, \\
\max & b^T y + dy_0 & \\
\text{s.t.:} & A^T y + y_0 a & \leq c.
\end{array}
$$

Let a basic feasible solution with basis x_B, x_N, A_B be defined in the usual way, so that when solving complementary slackness we have $A_B^T y + y_0 a = c_B$. Notice that when the linear dependence is exploited for some $\lambda \neq 0$,

$$
\begin{bmatrix} A^T \\ b^T \end{bmatrix} \lambda = \begin{bmatrix} a \\ d \end{bmatrix},
$$

we have from complementary slackness

$$
\begin{aligned}
A_B^{-T} y + y_0 (A_B^T \lambda) &= c_B \\
A_B^T (y + y_0 \lambda) &= c_B.
\end{aligned}
$$

Since A_B^T is invertible, we can pick any y_0^* to get a unique solution $y^* = A_B^{-T} c_B - y_0^* \lambda$. On the other hand, this produces the objective value

$$\begin{aligned} b^T(A_B^{-T} c_B - y_0^* \lambda) + d y_0^* &= (A_B^{-1} b)^T c_B - y_0^* b^T \lambda + d y_0 \\ &= (A_B^{-1} b)^T c_B, \end{aligned}$$

since $b^T \lambda = d$. We see that the particular choice of y_0^* doesn't matter, and can be chosen arbitrarily.