ORIE 6300 Mathematical Programming I	September 13, 2016
Lecture 7	
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1 Review

Last time we talked about normal cone of polyhedra:

$$N_{\{x|Ax\le b\}}(x) = \{A^T y | y \ge 0, y^T(b - Ax) = 0\}$$
$$N_{\{y|y\ge 0, A^T y = c\}}(y) = \{-b | (\exists x) Ax \le b, y^T(b - Ax) = 0\}$$

We also proved strong duality in feasible case.

2 Strong Duality (Continued)

Lemma 1 *Let $c \in \mathbb{R}^n$, suppose that $p^* = sup\ \{c^T x | x \in Q(A, b)\}$ is finite, then $\exists x^* \in Q(A, b)$, so that $p^* = c^T x^*$*

Proof: Let $S_1 = \{x | c^T x = p^*\}$ and $S_2 = Q(A, b),$. Suppose that $S_1 \cap S_2 = \emptyset$. We have the following claim:

Claim 2 *$S_1 - S_2 = \{x - \bar{x} | c^T x = p^*, A\bar{x} \le b\}$ is closed.*

We will prove this claim in the next lecture.

Thus by Separating Hyperplane Theorem:

$$\exists\ (\hat{a} \in \mathbb{R}^n \backslash \{0\}, \hat{b} \in \mathbb{R}),\ s.t. \sup_{x \in S_1} \hat{a}^T x < \hat{b} < \inf_{y \in S_2} \hat{a}^T y.$$

We know $\forall\ \varepsilon > 0, \exists\ x_\varepsilon \in S_2$, s.t. $p^* - \varepsilon \le c^T x_\varepsilon \le p^*$. Because $S_1 \subset \{x | \hat{a}^T x \le \hat{b}\}$, we have $\mathrm{dist}(x_\varepsilon, S_1) \ge \mathrm{dist}(x_\varepsilon, \{x | \hat{a}^T x \le \hat{b}\})$ (where $\mathrm{dist}(x, S) = \|x - P_S(x)\|$ denotes the distance of a point to a closed convex set S).

We leave following two conclusions as exercises ($x_+ := \max\{x, 0\}$)

1. $P_{S_1}(x) = x - \frac{c^T x - p^*}{\|c\|^2} c$;
2. $P_{\{x|\hat{a}^T x \le \hat{b}\}}(x) = x - \frac{(\hat{a}^T x - \hat{b})_+}{\|\hat{a}\|^2} \hat{a}$.

Thus

$$\begin{aligned} \mathrm{dist}(x_\varepsilon, S_1) &= \|x_\varepsilon - P_{S_1}(x_\varepsilon)\| \\ &= \|x_\varepsilon - \left(x_\varepsilon - \frac{c^T x - p^*}{\|c\|^2} c \right) \| \\ &= \frac{\|c^T x_\varepsilon - p^*\|}{\|c\|} \le \frac{\varepsilon}{\|c\|}, \end{aligned}$$

and, by a similar argument,

$$\begin{aligned}\text{dist}(x_\varepsilon, \{x|\hat{a}^T x \le \hat{b}\}) &= \|x_\varepsilon - P_{\{x|\hat{a}^T x \le \hat{b}\}}(x_\varepsilon)\| \\ &= \frac{|\hat{a}^T x_\varepsilon - b|}{\|\hat{a}\|}.\end{aligned}$$

By assumption, we have

$$0 < \inf_{x \in S_2} \frac{|\hat{a}^T x - \hat{b}|}{\|\hat{a}\|} \le \inf_{\varepsilon > 0} \frac{|\hat{a}^T x_\varepsilon - \hat{b}|}{\|\hat{a}\|} = \inf_{\varepsilon > 0} \text{dist}(x_\varepsilon, \{x|\hat{a}^T x \le \hat{b}\}) \le \inf_{\varepsilon > 0} \text{dist}(x_\varepsilon, S_1) \le \inf_{\varepsilon \ge 0} \frac{\varepsilon}{\|c\|} = 0$$

We have reached a contradiction, so

$$\exists x^* \in Q(A,b) \text{ s.t. } c^T x^* = p^*.$$

□

Theorem 3 *(Strong Duality Infeasible Case) Consider the linear programs:*

$$p^* = max(c^T x | Ax \le b),\ d^* = min(b^T y | A^T y = c, y \ge 0)$$

3. $d^* = \infty$ and primal is unbounded and $p^* = \infty$,
4. $p^* = -\infty$ and dual is unbounded and $d^* = -\infty$

Proof of 3:
(a) Suppose $d^* = \infty$ and the primal is feasible. If $\exists x^* \in Q(A,b)$, s.t. $c^T x^*$is maximal, then $c \in N_{Q(A,b)}(x^*) = \{A^T y | y \ge 0, y^T(b - Ax^*) = 0\}$. Any y that satisfies $c = A^T y, y \ge 0$ is feasible for the dual. This contradicts the infeasibility of the dual. Thus $p^* = \infty$ and $Q(A,b)$ is unbounded.
(b) If $Q(A,b)$ is unbounded, and $p^* = \infty$, then by weak dualiyty theorem, $d^* \ge p^* = \infty$ □
Proof of 4: We leave it as an exercise. □

3 Consequence of Strong Duality

Theorem 4 *(Theorem of Alternatives) Exactly one of the following hold:*

$$\begin{aligned}&1.\ \exists x,\ \ s.t.\ \ Ax \le b, \\ &2.\ \exists y,\ \ s.t.\ \ A^T y = 0, b^T y < 0, y \ge 0.\end{aligned}$$

Proof: We firstly prove 1 and 2 cannot hold simultaneously. If x satisfies 1 and y satisfies 2, then $0 = y^T(Ax) \le y^T b < 0$.
Secondly, suppose 1 is false, then $\max\{s | Ax + s\mathbf{1} \le b, s \le 0\}$ always has a solution (we are maximizing a negative number). Write this program in matrix form:

$$\begin{bmatrix} A & \mathbf{1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ s \end{bmatrix} \le \begin{bmatrix} b \\ 0 \end{bmatrix}$$

This linear program has an optimal solution (x^*, s^*) and optimal value $s^* < 0$. By strong duality, the dual

$$\begin{aligned} &\min\ b^T y \\ &\text{s.t} \quad \begin{bmatrix} A^T & 0 \\ \mathbf{1}^T & 1 \end{bmatrix} \begin{bmatrix} y \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &\qquad y \geq 0, t \geq 0 \end{aligned}$$

has an optimal solution (y^*, t^*) with optimal value $b^T y^* = s^* < 0$ and $A^T y^* = 0, y^* \geq 0$.
Now suppose 2 is false, then 1 cannot be false (otherwise 2 would be true), hence 1 is true. □

4 Sentivity Analysis and Value Function

Definition 1 *(Maximal Value Function)* $v(u) = \max\{c^T x | Ax \leq b + u\}$.

We have following two natural questions:

1. Can we bound the value function in terms of $v(0)$? If $v(u)$ is particularly expensive to compute, knowing a bound on it in terms of $v(0)$ can help us determine whether it might be worth it to re-solve the linear program.
2. What is the rate of change, i.e., the derivative of v?

Lemma 5 *Suppose $v(0)$ exists and $x^*(0) \in Q(A, b)$ satisfies $c^T x^*(0) = p^*$. Let $dom(v) := \{u | v(u) > -\infty\}$. Then following three hold:*

1. *$\forall u \in \mathbb{R}^m, v(u) < \infty$;*
2. *v is concave;*
3. *v is piecewise linear.*

Proof of 1: Since $x^*(0)$ exists, we know $\exists y_0 \geq 0$, s.t. $A^T y_0 = c$ (by strong duality), so $V(u) = \max\ \{c^T x | Ax \leq b + u\} \leq \min\{(b+u)^T y | y \geq 0, A^T y = c\} \leq (b+u)^T y_0 < \infty$ (by weak duality). □
Proof of 2: Let $u_1, u_2 \in \text{dom}(v)$ and let $\lambda \in [0, 1]$. By strong duality,

$$\begin{aligned} v(\lambda u_1 + (1-\lambda) u_2) &= \min\{(b + \lambda u_1 + (1-\lambda) u_2)^T y | y \geq 0, A^T y = c\} \\ &= \min\{\lambda (b + u_1)^T y + (1-\lambda)(b + u_2)^T y | y \geq 0, A^T y = c\} \\ &\geq \lambda \min\{(b + u_1)^T y | y \geq 0, A^T y = c\} + (1-\lambda) \min\{(b + u_2)^T y | y \geq 0, A^T y = c\} \\ &= \lambda v(u_1) + (1-\lambda) v(u_2). \end{aligned}$$

□

Proof of 3: By the results of Recitation 4, we have, for all $u \in \text{dom}(v)$,

$$\begin{aligned} v(u) &= \min\left\{(b+u)^T y | A^T y = c, y \geq 0\right\} \\ &= \min\left\{(b+u)^T y_k | y_1, \ldots, y_E \text{ are extreme points of } \{y | A^T y = c, y \geq 0\}\right\}. \end{aligned}$$

□