Lecture 12

Lecturer: Damek Davis *Scribe: Sergio Palomo*

1 Example of the Simplex Method

We introduced the simplex method in the last class. Consider a standard form LP and its dual:

$$
\begin{array}{ll}
\min & c^T x \\
\text{s.t.} & Ax = b \\
& x \geq 0
\end{array}
\qquad\qquad
\begin{array}{ll}
\max & b^T y \\
\text{s.t.} & A^T y \leq c
\end{array}
$$

Given a basic feasible solution x with corresponding basis B, we first compute the verifying $y = (A_B^T)^{-1} c_B$ and the reduced cost $\bar{c} = c - A^T y$. Recall that $\bar{c}_B = 0$, $x_N = 0$ and $x_B = A_B^{-1} b$. If $\bar{c} \geq 0$, x is an optimal solution. Otherwise, let $\bar{b} = A_B^{-1} b$, $\bar{A} = A_B^{-1} A_N$ and turn the original primal LP into the following equivalent problem:

$$
\begin{array}{ll}
\min & \bar{c}_N^T x_N \\
\text{s.t.} & \bar{A} x_N \leq \bar{b} \\
& x_N \geq 0
\end{array}
$$

Pick up $j \in N$ such that $\bar{c}_j < 0$ and do the following steps:

- Check for unboundedness: If $\bar{A}_{ij} \leq 0$ for all i, then the LP is unbounded.

- Ratio test: Compute
$$
\epsilon = \min_{i:\bar{A}_{ij} > 0} \frac{\bar{b}_i}{\bar{A}_{ij}}.
$$

  Let $i^* \in B$ be the i achieving the minimum. Construct a new basic feasible solution by setting $\hat{x}_j \leftarrow \epsilon$, $\hat{x}_k \leftarrow 0, \forall k \in N - \{j\}$, and $\hat{x}_B \leftarrow \bar{b} - \bar{A}\hat{x}_N$.

- Update basis: The new basis $\hat{B} \leftarrow B \cup \{j\} - \{i^*\}$.

Now a better basic feasible solution $\hat{x}$ with associated basis $\hat{B}$ is generated. By iterating through the steps above, we will finally find an optimal solution or assert that the problem is actually unbounded. In some sense, the simplex method is a local search. We only swap one basic column and non-basic column during each iteration.

Here is a concrete example of simplex method.

Example 1 *Solve the following LP:*

$$
\begin{array}{rrrrrrr}
min & -x_1 & + & 2x_2 & - & x_3 & \\
s.t. & x_1 & & & & & \leq 4 \\
& & & x_2 & & & \leq 4 \\
& x_1 & + & x_2 & & & \leq 6 \\
& -x_1 & & & + & 2x_3 & \leq 4 \\
& \multicolumn{5}{c}{x_1, x_2, x_3 \geq 0}
\end{array}
$$

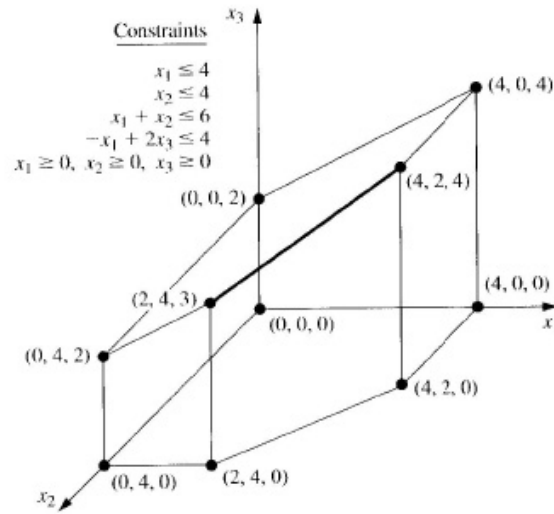*The feasible region of this problem is visualized in Figure 1.*

Figure 1: 3-D Diagram of the Feasible Region

To apply the simplex method, convert it to a standard form problem by adding slack variables:

$$
\begin{array}{rlllll}
\min & -x_1 & + & 2x_2 & - & x_3 \\
\text{s.t.} & x_1 & & & + & x_4 & & & = 4 \\
& & x_2 & & & + & x_5 & & = 4 \\
& x_1 & + & x_2 & & & + & x_6 & = 6 \\
& -x_1 & & + & 2x_3 & & & + & x_7 = 4 \\
& & & x_1, \ldots, x_7 \geq 0
\end{array}
$$

Rewrite the constraints in matrix form:

$$
\begin{bmatrix}
1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 1 & 0 \\
-1 & 0 & 2 & 0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
x_1 \\ x_2 \\ \vdots \\ x_7
\end{bmatrix}
=
\begin{bmatrix}
4 \\ 4 \\ 6 \\ 4
\end{bmatrix}
$$

$$
x \geq 0
$$

To start the simplex method, an initial basic feasible solution is needed. Since the last four columns in the coefficient matrix form an identity matrix, it is convenient to regard them as a basis first and check the feasibility of the basic solution corresponding to it. Now B refers to last four columns, N refers to first three columns, and

$$
A_B = I, x_N = 0, x_B = A_B^{-1} b =
\begin{bmatrix}
4 \\ 4 \\ 6 \\ 4
\end{bmatrix}.
$$

It is a feasible solution so we could begin the iteration based on it.

Compute

$$y = (A_B^T)^{-1} c_B = I \cdot 0 = 0, \bar{c} = c - A^T y = \begin{bmatrix} -1 \\ 2 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

x is not optimal because $\bar{c} \not\geq 0$. Since $\bar{c}_1 < 0$, increasing x_1 will decrease the objective function. Now

$$\bar{b} = A_B^{-1} b = \begin{bmatrix} 4 \\ 4 \\ 6 \\ 4 \end{bmatrix}, \bar{A}_1 = A_B^{-1} A_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \end{bmatrix}.$$

So the problem is not unbounded at this point. From ratio test $\epsilon = 4$, we can increase x_1 from 0 to 4 without violating the constraints. At the same time x_4 becomes 0, so 1 enters the basis and 4 leaves it. Now B refers to column $\{1, 5, 6, 7\}$ and N refers to column $\{2, 3, 4\}$.

If $A_B = I$ is maintained in the next iteration, our calculation would be greatly simplified. By doing elementary row operations, we could get a unit vector in the first column while keeping the feasible region of the original problem untouched. The equivalent new constraints are

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 & 1 \end{bmatrix} x = \begin{bmatrix} 4 \\ 4 \\ 2 \\ 8 \end{bmatrix}.$$

Our new basic feasible solution is $\begin{bmatrix} 4 & 0 & 0 & 0 & 4 & 2 & 8 \end{bmatrix}^T$.

To simplify the problem further, we wish the objective function only contain non-basic variables to make $c_B = 0$. By substituting the first constraint $x_1 = 4 - x_4$ into the objective function, it becomes $-4 + 2x_2 - x_3 + x_4$.

Now compute

$$y = (A_B^T)^{-1} c_B = I \cdot 0 = 0, \bar{c} = c - A^T y = \begin{bmatrix} 0 \\ 2 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

x is not optimal because $\bar{c} \not\geq 0$. Since $\bar{c}_3 < 0$, increasing x_3 will decrease the objective function. Now

$$\bar{b} = A_B^{-1} b = \begin{bmatrix} 4 \\ 4 \\ 2 \\ 8 \end{bmatrix}, \bar{A}_3 = A_B^{-1} A_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \end{bmatrix}.$$

So the problem is not unbounded at this point. From ratio test $\epsilon = 4$, we could increase x_3 from 0 to 4 without violating the constraints. At the same time x_7 becomes 0, so 3 enters the basis and 7 leaves it. Now B refers to column $\{1, 3, 5, 6\}$ and N refers to column $\{2, 4, 7\}$.

Diving last row by 2 to keep $A_B = I$, we get

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1/2 & 0 & 0 & 1/2 \end{bmatrix} x = \begin{bmatrix} 4 \\ 4 \\ 2 \\ 4 \end{bmatrix}.$$

Our new basic feasible solution is $\begin{bmatrix} 4 & 0 & 4 & 0 & 4 & 2 & 0 \end{bmatrix}^T$.

Recall our old objective function $-4 + 2x_2 - x_3 + x_4$. Since x_3 is now in the basis, the corresponding coefficient needs to be zero. Substitute $x_3 = 4 - 1/2x_4 - 1/2x_7$ (which we get from the fourth constraint) into the objective function. From this we get our new (reduced) objective function $-8 + 2x_2 + 3/2x_4 + 1/2x_7$.

Since $\bar{c} \geq 0$ we are finished with simplex. The optimal solution x^* is $\begin{bmatrix} 4 & 0 & 4 & 0 & 4 & 2 & 0 \end{bmatrix}^T$.

2   Finding an Initial Basic Feasible Solution

We now start addressing various issues with the simplex method. The first question is how we get an initial basic feasible solution with associated basis. The idea for achieving this is to modify the LP such that there is an easy initial basic feasible solution, and solving the modified LP gives some (not necessarily optimal) basic feasible solution to the original LP.

Without loss of generality we can assume $b \geq 0$, since if some $b_i < 0$, we multiply that constraint by -1 (this is okay since there are only equality constraints). Consider the following modification to the original problem

$$\begin{aligned} \min \quad & e^T z \\ \text{s.t.} \quad & Ax + Iz = b \\ & x \geq 0 \\ & z \geq 0 \end{aligned}$$

where $e = \begin{bmatrix} 1 & 1 & \dots & 1 \end{bmatrix}^T$ is a column of ones. The z variables are called *artificial variables*, and the x's are called *real variables*. Define

$$A' = \begin{bmatrix} A \mid I \end{bmatrix}, x' = \begin{bmatrix} x \\ z \end{bmatrix}$$

so the constraints of the modified LP can be written as $A'x' = b$, $x' \geq 0$.

Now we could run the simplex method on this modified LP, starting with the basis corresponding to z variables. Then $x = 0, z = b \geq 0$ is a basic feasible solution.

The simplex method can have one of two possible results (note that the modified LP is never unbounded: since $z \geq 0$, the objective function is bounded from below by 0).

- Case 1: The value of modified LP is non-zero. Then there are no feasible solutions to the original LP.

- Case 2: The value of modified LP is zero. If we find a solution (x, z) such that $e^T z = 0$. Then x is feasible for the original LP. We still need to ensure that x is actually a basic solution to the original LP. There are two subcases:

  - The Good Case: All artificial variables are non-basic. Then $A'_B = A_B$, so that $x'_B = (A'_B)^{-1}b$, $x'_N = 0$ is feasible for $A'x' = b$, so $x_B = A_B^{-1}b$, $x_N = 0$ is a basic feasible solution for $Ax = b$. We can now run the simplex method for the original LP, starting with the basis B.

  - The Bad Case: Some artificial variables are in the basis. The idea to deal with this case is to swap the artificial variables out of the basis. We will talk about it next time.