1 Last Time:

Theorem 1 (KM Theorem) *Suppose $N : \mathbb{R}^n \to \mathbb{R}^n$ is 1-Lipschitz continuous, i.e. $(\forall x \in R^n)(\forall y \in \mathbb{R}^n)\|Nx - Ny\| \leq \|x - y\|$, that $\mathrm{Fix}(N) \neq \emptyset$, and $\lambda \in (0, 1)$. Then, given any $z^0 \in \mathbb{R}^n$ the sequence $\{z^k\}_{k\in\mathbb{N}}$ generated by the KM iteration*

$$z^{k+1} = N_\lambda z^k = (1 - \lambda)z^k + \lambda N z^k$$

converges to an element of $\mathrm{Fix}(N)$.

2 The Method of Alternating Projections (MAP)

Suppose

$$x^* \in \operatorname{argmin}\left\{c^T x \mid Ax = b, x \geq 0\right\} \quad \text{and} \quad (y^*, s^*) \in \operatorname{argmax}\left\{b^T y \mid A^T y + s = c, s \geq 0\right\}.$$

Using strong duality, these inclusions are equivalent to

$$Ax^* = b; \quad A^T y^* + s = c; \quad c^T x^* - b^T y^* = 0; \quad x^* \geq 0; \quad s^* \geq 0.$$

Define the set C_1 as the solutions to

$$\begin{bmatrix} A & 0 & 0 \\ 0 & A^T & 1 \\ c^T & -b^T & 0 \end{bmatrix} \begin{bmatrix} x^* \\ y^* \\ s^* \end{bmatrix} = \begin{bmatrix} b \\ c \\ 0 \end{bmatrix}$$

and the set $C_2 = \{(x, y, s) \in \mathbb{R}^{m+2n} \mid x, s \geq 0\}$. We have just shown that LPs can actually be cast as a **feasbility problem**:

Theorem 2 *The pair (x^*, y^*) is primal-dual optimal if, and only if, there exists $s^* \in \mathbb{R}_{\geq 0}$ such that $(x^*, y^*, s^*) \in C_1 \cap C_2$.*

Now, let's take a step back and consider two closed, convex sets $C_1, C_2 \subseteq \mathbb{R}^n$. Let's solve, $x \in C_1 \cap C_2$ by forming an operator $N : \mathbb{R}^n \to \mathbb{R}^n$ with fixed points $C_1 \cap C_2$. To apply the KM theorem, the operator N must be 1-Lipschitz continuous.

Definition 1 *We call a 1-Lipschitz mapping $N : \mathbb{R}^n \to \mathbb{R}^n$* ***nonexpansive****.*

We will often find the following identity useful:

Lemma 3 *For all $a, b \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$, we have*

$$\|(1-\lambda)a + \lambda b\|^2 = (1-\lambda)\|a\|^2 + \lambda\|b\|^2 - \lambda(1-\lambda)\|a-b\|^2.$$

Before we construct the operator N, we prove a Lemma which shows that projection mappings satisfy a property slightly stronger than nonexpansiveness.

Lemma 4 *Let $C \subseteq \mathbb{R}^n$ be a closed convex set. Then $(\forall x \in \mathbb{R}^n)(\forall y \in \mathbb{R}^n)$*

$$\|P_C(x) - P_C(y)\|^2 \leq \|x-y\|^2 - \|(x - P_C(x)) - (y - P_C(y))\|, \quad \text{(firm non-expansiveness)}$$

In particular, P_C and $2P_C - I$ are nonexpansive.

Proof: Recall that $x - P_C(x) \in N_C(P_C(x))$ and $y - P_C(y) \in N_C(P_C(y))$, so

$$\langle x - P_C(x), P_C(y) - P_C(x)\rangle \leq 0 \quad \text{and} \quad \langle y - P_C(y), P_C(x) - P_C(y)\rangle \leq 0.$$

Add these inequalities to get

$$\begin{aligned}
0 &\geq \langle (x - P_C(x)) - (y - P_C(y)), P_C(y) - P_C(x)\rangle, \\
&= \frac{1}{2}\left(-\|x-y\|^2 + \|(x - P_C(x)) - (y - P_C(y))\|^2 + \|P_C(x) - P_C(y)\|^2\right), \quad \text{law of cosines} \\
&\iff \|P_C(x) - P_C(y)\|^2 \leq \|x-y\|^2 - \|(x - P_C(x)) - (y - P_C(y))\|^2.
\end{aligned}$$

Thus P_C is firmly nonexpansive. Finally, by Lemma 3, we have

$$\begin{aligned}
&\|(2P_C(x) - x) - (2P_C(y) - y)\|^2 = \|2(P_C(x) - P_C(y)) + (1-2)(x-y)\|^2, \\
&\quad = 2\|P_C(x) - P_C(y)\|^2 + (1-2)\|x-y\|^2 - 2(1-2)\|(P_C(x) - x) - (P_C(y) - y)\|^2, \\
&\quad \leq 2\left[\|x-y\|^2 - \|(x - P_C(x)) - (y - P_C(y))\|^2\right] - \|x-y\|^2 + 2\|(x - P_C(x)) - (y - P_C(y))\|^2, \\
&\quad = \|x-y\|^2.
\end{aligned}$$

□

Corollary 5 *Let $C_1, C_2 \subseteq \mathbb{R}^m$ be closed nonempty convex sets. Then $N = \frac{3}{2}P_{C_2}P_{C_1} - \frac{1}{2}I$ is nonexpansive.*

Proof: Recall that $\|\cdot\|^2$ is a convex function, so

$$\left\|\frac{1}{2}(x+y)\right\|^2 \leq \frac{1}{2}\|x\|^2 + \frac{1}{2}\|y\|^2, \quad \text{i.e.} \quad \frac{1}{2}\|x+y\|^2 \leq \|x\|^2 + \|y\|^2.$$

Now let $x, y \in \mathbb{R}^n$.

$$\begin{aligned}
&\frac{1}{2}\|(I - P_{C_2}P_{C_1})(x) - (I - P_{C_2}P_{C_1})(y)\|^2 \\
&\quad = \frac{1}{2}\|(I - P_{C_1})(x) - (I - P_{C_1})(y) + (P_{C_1} - P_{C_2}P_{C_1})(x) - (P_{C_1} - P_{C_2}P_{C_1})(y)\|^2, \\
&\quad \leq \|(I - P_{C_1})(x) - (I - P_{C_1})(y)\|^2 + \|(P_{C_1} - P_{C_2}P_{C_1})(x) - (P_{C_1} - P_{C_2}P_{C_1})(y)\|^2, \\
&\quad \leq \|x-y\|^2 - \|P_{C_1}(x) - P_{C_1}(y)\|^2 + \|P_{C_1}(x) - P_{C_1}(y)\|^2 - \|P_{C_2}P_{C_1}(x) - P_{C_2}P_{C_1}(y)\|^2,
\end{aligned}$$

where we apply Lemma 4 twice to get the last inequality. Thus,

$$\|P_{C_2}P_{C_1}(x) - P_{C_2}P_{C_1}(y)\|^2 + \frac{1}{2}\|(I - P_{C_2}P_{C_1})(x) - (I - P_{C_2}P_{C_1})(y)\|^2 \leq \|x - y\|^2.$$

Therefore, by Lemma 3, we have

$$\begin{aligned}
&\|N(x) - N(y)\|^2 = \left\|\frac{3}{2}(P_{C_2}P_{C_1}(x) - P_{C_2}P_{C_1}(y)) - \frac{1}{2}(x - y)\right\|^2, \\
&= \frac{3}{2}\|P_{C_2}P_{C_1}(x) - P_{C_2}P_{C_1}(y)\|^2 - \frac{1}{2}\|x - y\|^2 + \frac{3}{4}\|(I - P_{C_2}P_{C_1})(x) - (I - P_{C_2}P_{C_1})(y)\|^2, \\
&\leq \frac{1}{2}\left[3\|x - y\|^2 - \|x - y\|^3\right], \\
&= \|x - y\|^2.
\end{aligned}$$

□

We'll use the following simple fact.

Exercise 1 *Fix*$(P_C) = C$.

Proposition 6 *Let $C_1, C_2 \subseteq \mathbb{R}^n$ be closed, convex sets such that $C_1 \cap C_2 \neq \emptyset$. Then*

$$C_1 \cap C_2 = Fix(P_{C_2} \circ P_{C_1}) = Fix\left(\frac{3}{2}P_{C_2} \circ P_{C_1} - \frac{1}{2}I\right).$$

Proof: $N = P_{C_2} \circ P_{C_2}$ is nonexpansive by Lemma 4, and the fact that the compositions of nonexpansive maps are nonexpansive. Now, let $x \in C_1 \cap C_2$. Then $P_{C_1}(x) = x$ and $P_{C_2}(x) = x$. Thus, $(P_{C_2} \circ P_{C_1})(x) = x$ and $x \in \text{Fix}(N)$.

Now suppose, $x \in \text{Fix}(N)$. Then

$$x = P_{C_2} \circ P_{C_1}(x),$$

and so $x \in C_2$. We consider three cases:

1. Suppose $P_{C_1}(x) \in C_2$. Then $x = P_{C_2}P_{C_1}x = P_{C_1}x$, so $x \in C_1 \cap C_2$.
2. Suppose $x \in C_1$. Then $x \in C_1 \cap C_2$.
3. Suppose $x \notin C_1$ and $P_{C_1}x \notin C_2$. Then $\forall y \in C_1 \cap C_2$, we have

$$\begin{aligned}
\|x - y\| &= \|P_{C_2}P_{C_1}(x) - P_{C_2}P_{C_1}(y)\|, \\
&< \|P_{C_1}(x) - P_{C_1}(y)\|, \quad (P_{C_1}(y) = y \in C_2 \quad \text{and} \quad P_{C_1}(x) \notin C_2, \\
&< \|x - y\|, \quad (x \notin C_1 \quad \text{and} \quad y \in C_1 \cap C_2).
\end{aligned}$$

This is a contradiction! So $x \in C_1 \cap C_2$.

The equality $\text{Fix}(P_{C_2}P_{C_1}) = \text{Fix}(\frac{3}{2}P_{C_2}P_{C_1} - \frac{1}{2}I)$ follows because $P_{C_2}P_{C_1} = \frac{2}{3}\left(\frac{3}{2}P_{C_2}P_{C_1} + (1 - \frac{3}{2})I\right) + \frac{1}{3}I$.

□

Theorem 7 *Suppose $C_1, C_2 \subseteq \mathbb{R}^n$ are closed convex sets such that $C_1 \cap C_2 \notin \emptyset$. Let $z^0 \in \mathbb{R}^n$. Then the Method of Alternating Projections*

$$z^{k+1} = P_{C_2} P_{C_1} z^k$$

converges to an element of $C_1 \cap C_2$.

Proof: Let $N = \frac{3}{2} P_{C_2} P_{C_1} - \frac{1}{2} I$, apply KM iteration theorem with $\lambda = \frac{2}{3}$ and observe that

$$N_\lambda = (1-\lambda)I + \lambda N = \frac{1}{3} I + P_{C_2} P_{C_1} - \frac{1}{3} I = P_{C_2} P_{C_1}.$$

□

Remark 1 *1. In general, the method of alternating projections can converge arbitrarily slowly!*

2. If $C_1 \cap C_2 = \emptyset$, then under certain conditions

$$\|z^k - P_{C_1}(z^k)\| \to \inf_{z \in C_2, w \in C_1} \|z - w\|$$

and $z^k - P_{C_1}(z^k)$ converges to the gap vector $v = z^* - w^*$, where $(z^*, w^*) \in \operatorname{argmin}_{z \in C_2, w \in C_1} \|z - w\|$

3. Was originally introduced by van-Neumann and Halperin in the 1930s.

Returning to the LP feasibility problem, i.e.: we let C_1 be the solutions to

$$\begin{bmatrix} A & 0 & 0 \\ 0 & A^T & 1 \\ c^T & -b^T & 0 \end{bmatrix} \begin{bmatrix} x^* \\ y^* \\ s^* \end{bmatrix} = \begin{bmatrix} b \\ c \\ 0 \end{bmatrix}$$

and the set $C_2 = \{(x, y, s) \in \mathbb{R}^{m+2n} \mid x, s \geq 0\}$. Then we consider the feasibility problem:

$$\begin{bmatrix} x^* \\ y^* \\ s^* \end{bmatrix} \in C_1 \cap C_2.$$

Let's apply the MAP algorithm. Must compute projections first. Let $z = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

- The projection onto C_2 is a simple thresholding operation:

$$P_{C_2}(z) = \begin{bmatrix} \max\{x, 0\} \\ y \\ \max\{s, 0\} \end{bmatrix}.$$

- Computing $P_{C_1}(z)$ requires a linear system solve. Let

$$D = \begin{bmatrix} A & 0 & 0 \\ 0 & A^T & 1 \\ c^T & -b^T & 0 \end{bmatrix}$$

then

$$P_{C_1}(z) = z - D^\dagger \left(Dz - \begin{bmatrix} b \\ c \end{bmatrix} \right),$$

where $D^\dagger$ is the Moore-Penrose inverse. When D has full rank,

$$P_{C_1}(z) = z - D^\dagger \left(DD^\dagger \right)^{-1} \left(Dz - \begin{bmatrix} b \\ c \end{bmatrix} \right).$$

- The matrix $D^\dagger$ can be computed offline or one can solve the equation at each iteration. If one intends to run the algorithm for a long time, it may be a good idea to precompute $D^\dagger$.

- Furthermore, it can be shown that, given $z^0 = \begin{bmatrix} x^0 \\ y^0 \\ s^0 \end{bmatrix}$,

$$z^{k+1} = P_{C_2} P_{C_1}(z^k),$$

the MAP sequence converges *linearly*.

Theorem 8 *There exists* $\delta \in (0,1)$ *such that for all* $k \in \mathbb{N}$

$$\text{dist}_{C_1 \cap C_2}(z^{k+1}) \leq \delta \text{dist}_{C_1 \cap C_2}(z^k).$$

Hence, for all $k \in \mathbb{N}$ $\text{dist}_{C_1 \cap C_2}(z^k) \leq \delta^k \text{dist}_{C_1 \cap C_2}(z^0)$.

In general, δ depends on the "angle" between C_1 and C_2. The more transversely they meet, the better.