ORIE 6300 Mathematical Programming I November 17, 2016

Problem Set 10

Due Date: December 2, 2016

1. **Subdifferential Examples.** Prove the following
   (a) **ℓ_p norms.** $g(x) = \|x\|_p$ where $p \in [1, \infty]$. Let q satisfy $1/p + 1/q = 1$. Show that
   $$\partial g(0) = \{v \mid \|v\|_q \leq 1\}.$$
   (b) **Least Absolute Deviation (LAD).** $g(x) = \sum_{i=1}^m |\langle a_i, x\rangle - b_i|$ where $a_i \in \mathbb{R}^n$ and $b_i \in \mathbb{R}$. Show that
   $$\partial g(x) = \sum_{i \in I_+(x)} a_i - \sum_{i \in I_-(x)} a_i + \sum_{i \in I_0(x)} a_i * [-1, 1],$$
   where $a_i * [-1, 1] = \{a_i \lambda \mid \lambda \in [-1, 1]\}$ and
   $I_+(x) = \{i \mid \langle a_i, x\rangle - b_i > 0\}, \quad I_-(x) = \{i \mid \langle a_i, x\rangle - b_i < 0\}, \quad I_0(x) = \{i \mid \langle a_i, x\rangle - b_i = 0\}.$
   Using this decomposition, compute $[\partial\|\cdot\|_1](x)$.
   (c) **Distance Functions.** Let $C \subseteq \mathbb{R}^n$ be a closed, convex set. Let
   $$(\forall x \in \mathbb{R}^n) \qquad d_C(x) := \inf_{y \in C} \|y - x\|.$$
   Prove that d_C is closed, convex and 1-Lipschitz continuous. Suppose that $x \notin C$. Show that
   $$\frac{1}{d_C(x)}(x - P_C(x)) \in \partial d_C(x).$$
2. **Support Function/Normal Cone Duality.** Suppose that $C \subseteq \mathbb{R}^n$ is a closed convex set. Let $\sigma_C : \mathbb{R}^n \to \overline{\mathbb{R}}$ be the support function of C:
   $$(\forall v \in \mathbb{R}^n) \qquad \sigma_C(v) := \sup_{y \in C} \langle y, v\rangle.$$
   (a) Show that σ_C is closed and convex.
   (b) Show that $y \in \partial\sigma_C(v) \iff (y \in C \text{ and } \langle v, y\rangle = \sigma_C(v)) \iff v \in N_C(y)$.
3. **Lipschitz Continuity and Subdifferentials.** Prove that a closed convex function $g : \mathbb{R}^n \to \overline{\mathbb{R}}$ is L-Lipschitz continuous if, and only if, $\text{dom}(g) = \mathbb{R}^n$ and
   $$(\forall x \in \mathbb{R}^n) \qquad v \in \partial g(x) \implies \|v\| \leq L.$$
   Use this fact to conclude that the value function
   $$v(u) := \max\{c^T x \mid Ax \leq b + u\}$$
   is L-Lipschitz continuous if, and only if, the dual polytope $P(A^T, c) = \{y \mid A^T y = c, y \geq 0\}$ is nonempty and bounded.

4. **Subgradient Method.** Let $g : \mathbb{R}^n \to \mathbb{R}$ be an L-Lipschitz continuous convex function, let $C \subseteq \mathbb{R}^n$ be a closed, convex set, and suppose that $x^* \in \operatorname{argmin}_{x \in C} g(x)$.

   (a) Let $\gamma > 0$, let $x \in \mathbb{R}^n$, let $v \in \partial g(x)$, and define

   $$x_{\gamma,v} = P_C(x - \gamma v)$$

   Prove that

   $$\|x_{\gamma,v} - x^*\|^2 + 2\gamma(g(x) - g(x^*)) \leq \|x - x^*\|^2 + \gamma^2 L^2.$$

   (**Hint:** First assume that $C = \mathbb{R}^n$, then use nonexpansiveness of P_C.)

   (b) Consider the subgradient descent method

Algorithm 1 Projected Subgradient Method for $\operatorname{argmin}_{x \in C} g(x)$

Input: $x^0 \in C, \{\gamma_k\}_{k \in \mathbb{N}} \subseteq \mathbb{R}_{>0}$

loop
    Choose $v^k \in \partial g(x^k)$
    $x^{k+1} = P_C(x^k - \gamma_k v^k)$

   Prove that

   $$(\forall k \in \mathbb{N}) \qquad \min_{i=0,\ldots,k} \left\{ g(x^i) - g(x^*) \right\} \leq \frac{\|x^0 - x^*\|^2 + L^2 \sum_{i=0}^k \gamma_i^2}{2 \sum_{i=0}^k \gamma_i}$$

   Give a choice of $\{\gamma_k\}_{k \in \mathbb{N}}$ that guarantees $\min_{i=0,\ldots,k} \left\{ g(x^i) - g(x^*) \right\} \to 0$ as $k \to \infty$.