ORIE 6300 Mathematical Programming I September 1, 2016

Problem Set 2

Due Date: September 8, 2016

1. Give an example of a primal-dual pair for which both the primal and dual are infeasible, and demonstrate that they are infeasible. Use a matrix $A \in \Re^{1\times 1}$.

2. Let $\mathbf{1} \in \mathbb{R}^n$ be the vector of all ones. Consider the set of doubly stochastic matrices

$$X = \{A \in \mathbb{R}^{n\times n} \mid A \geq 0 \text{ (entrywise)}, \mathbf{1}^T A = \mathbf{1}^T \text{ and } A\mathbf{1} = \mathbf{1}\}.$$

   Prove that X is convex and a polytope. Show that the set of extreme points of X is exactly the set of permutation matrices $\mathcal{P}$, i.e., those binary matrices $P \in \mathbb{R}^{n\times n}$ that have exactly one entry equal to 1 in each row and each column and 0s elsewhere.

   (Hint: You can assume the following Lemma: Consider the polyhedron $Q := \{x \mid x \geq 0, Cx = d\}$. Then

   - $\overline{x} \in Q$ is an extreme point if

$$\text{rank}\left(\begin{bmatrix} c_{i_1} & c_{i_2} & \dots & c_{i_k}\end{bmatrix}\right) = k$$

     where c_j is column j of C and $\{i_1, i_2, \dots, i_k\} = \{i \mid \overline{x}_i > 0\}$.
   - any extreme point of Q has at most rank(C) nonzero elements.

   Bonus points: prove the lemma.)

3. Let $C \subseteq \mathbb{R}^n$ be a closed convex set and let $y \in \mathbb{R}^n$ be a vector.

   (a) Show that $f : C \to \mathbb{R}^n$, defined by

$$(\forall x \in C) \quad f(x) = \frac{1}{2}\|x - y\|^2$$

   has a unique minimizer $x^* \in C$. (Hint: recall that in lecture 4 we showed f has at least one minimizer in C.)

   (b) Show that

$$(\forall z \in C) \qquad \|x^* - z\|^2 + \|x^* - y\|^2 \leq \|y - z\|^2.$$

   (Hint: if $y \in C$, the result is trivial; if $y \notin C$, recall from the proof of the separating hyperplane theorem, we had, for some $b \in \mathbb{R}$, that $(\forall z \in C)\ (y - x)^T z < b < (y - x)^T y$.)

   (c) Conclude that the *projection mapping* $P_C : \mathbb{R}^n \to \mathbb{R}^n$, defined by

$$(\forall y \in \mathbb{R}^n) \qquad P_C(y) = \text{the unique minimizer } x^* \in C \text{ of } f(x) = \frac{1}{2}\|x - y\|^2,$$

   is well-defined and

$$(\forall z \in C), (\forall y \in \mathbb{R}^n) \qquad \|P_C(y) - z\|^2 + \|P_C(y) - y\|^2 \leq \|y - z\|^2.$$

4. Suppose that you are given a feasible solution $\bar{x}$ of value $\bar{\gamma}$ to the problem $\max(c^T x : Ax \leq b)$. Give a method that either demonstrates that the feasible region is unbounded (i.e., there is a point x and direction y such that $x + \lambda y$ is feasible for all $\lambda > 0$) or that finds a vertex $\tilde{x}$ of the feasible region with objective value $c\tilde{x} \geq \bar{\gamma}$. Your method should not use general purpose linear programming algorithms (like the simplex method). (Hint: some of the discussion of the equivalence of bounded polyhedra and polytopes, as well as the equivalence of extreme points, vertices, and basic feasible solutions, might prove useful.)