ORIE 6300 Mathematical Programming I September 15, 2016

Problem Set 4

Due Date: September 22, 2016

1. Compute the projection operator P_S for each of the following closed, convex sets S:
   (a) $S = \{x \in \mathbb{R}^n \mid x \geq 0\}$.
   (b) $S = [-1, 1]^n$.
   (c) $S = \{x \mid Ax = b\}$ where $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$.
   (d) $S = \{x \mid a^T x \leq b\}$ where $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$.

2. Consider the set $P = \{x : Ax \geq 0\}$ and assume that we have $x \geq 0$ for all $x \in P$, i.e., that $x \geq 0$ is implied by $Ax \geq 0$.
   (a) A set K is a cone if $x, y \in K$ implies that $\lambda x + \mu y \in K$ for all $\mu, \lambda \geq 0$. Prove that P is a cone.
   (b) An *extreme ray* of a cone K is a nonzero vector $x \in K$ such that $x + y \in K$ and $x - y \in K$ implies that $y = \lambda x$ for some λ.
   Give another characterization of the extreme rays of the polyhedral cone P, using the rank of a submatrix of A. (Hint: think about the positive orthant as the canonical example of a cone, in order to get some intuition here.)
   (c) Two extreme rays x and y of a cone K are said to be the same if $x = \lambda y$ for some $\lambda > 0$. Prove that the number of different extreme rays of our polyhedral cone P is finite. Give a finite bound on the maximum number of extreme rays possible assuming that A is has m rows and n columns.
   (d) Let $r^1, \ldots, r^k$ denote the finite set of extreme rays of P. Let
   $$Q = cone(r^1, \ldots, r^k) = \{x = \sum_i \lambda_i r^i : \lambda_i \geq 0 \text{ for all } i\}.$$
   Prove that $P = Q$. (Hint: consider $P' = \{x \in P : \sum x_i = 1\}$.)
   It might help to visualize this as moving from the description of P by the faces of the cone that bound it ($Ax \geq 0$) to a description of P by the outside rays ($r^1, \ldots, r^k$) that bound it.

3. (Strict Complementary Slackness) Consider the standard form linear programs, with primal LP ($\min c^T x : Ax = b, x \geq 0$) and dual LP ($\max b^T y : A^T y \leq c$). Suppose the value of the two LPs is γ.
   (a) Show that the set of optimal solutions to the primal is a convex set; argue the same for the dual.

(b) Show that either there exists an optimal solution x to the primal such that $x_j > 0$ or there exists an optimal solution y to the dual such that the jth inequality is strict; that is, $\sum_{i=1}^{n} a_{ij} y_i < c_j$. (Hint: Consider the linear program $(\min -e_j^T x : Ax = b, -c^T x \geq -\gamma, x \geq 0)$, where e^j is a vector that has a 1 in the jth component, and 0 everywhere else).

(c) Show that there exist a primal optimal solution x^* and a dual optimal solution y^* such that for all j, $x_j^* > 0$ if and only if the jth inequality of the dual is met with equality.